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**A SADDLEPOINT THEOREM FOR SELF-DUAL  
LINEAR SYSTEMS**

**Marjorie L. Stein**

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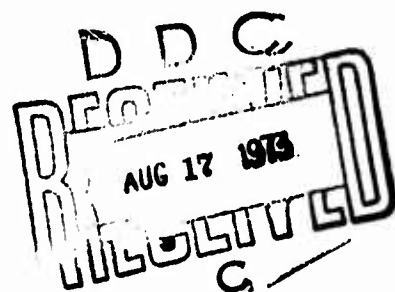


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MRC Technical Summary Report #1311  
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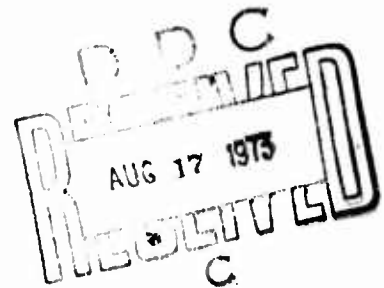
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### ABSTRACT

Let  $A$  be a skew matrix of order  $n$  over an ordered field. There is a finite class of skew matrices  $\bar{A}$  such that  $XA = Y$  and  $\bar{X}\bar{A} = \bar{Y}$  have the same solution sets, where  $\bar{x}_i = y_i$  and  $\bar{y}_i = x_i$  for some indices  $i$  (perhaps none) and  $\bar{x}_i = x_i$ ,  $\bar{y}_i = y_i$  for the remaining  $i$ . We show that for each index  $h$ ,  $1 \leq h \leq n$ , there exists an  $\bar{A}$  such that  $\bar{a}_{hj} \geq 0$  for all  $j$ . A proof of von Neumann's Minimax Theorem for symmetric games is one application of the theory.

# A SADDLEPOINT THEOREM FOR SELF-DUAL LINEAR SYSTEMS\*

Marjorie L. Stein

A finite class of skew coefficient matrices over an ordered field can be associated with a self-dual linear system. Pivot techniques allow us to transform a given matrix of the class into another while maintaining the same sets of solutions. The exchange of two independent variables in the system for two corresponding dependent ones maintains skewsymmetry.

Our main theorem says that in a finite number of pivot steps we can transform any skew matrix into a skew matrix with a saddlepoint — that is, a non-negative row and corresponding nonpositive column. A tableau form introduced by A. W. Tucker permits an immediate translation of this result about matrices into a statement about types of solutions that must occur in the associated self-dual linear system. This theorem and its corollaries yield an elementary proof of the Minimax Theorem for symmetric games.

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\* This paper contains portions of the author's doctoral dissertation [9], written at Princeton University under the supervision of Professor A. W. Tucker.

# §1. Pivoting in Skew Matrices

Dual systems of linear equations can be represented by the following tableau form of A. W. Tucker [12]:

$$\begin{array}{c}
 \begin{array}{ccc}
 & v_1 & \dots & v_n \\
 x_1 & \boxed{a_{11} \dots a_{1n}} & = -u_1 \\
 \vdots & & \vdots \\
 x_m & \boxed{a_{m1} \dots a_{mn}} & = -u_m \\
 & = y_1 \dots = y_n
 \end{array}
 & \text{or} &
 \begin{array}{ccc}
 & V \\
 X & \boxed{M} & = -U \\
 & = Y
 \end{array}
 \end{array}$$

Here  $M$  is an  $m \times n$  matrix,  $X$  is  $1 \times m$ ,  $Y$  is  $1 \times n$ ,  $U$  is  $m \times 1$  and  $V$  is  $n \times 1$ . This format expresses both a column system:

$$\left. \begin{array}{l}
 x_1 a_{11} + \dots + x_m a_{m1} = y_1 \\
 \vdots \\
 x_1 a_{1n} + \dots + x_m a_{mn} = y_n
 \end{array} \right\} XM = Y$$

and a row system:

$$\left. \begin{array}{l}
 a_{11} v_1 + \dots + a_{1n} v_n = -u_1 \\
 \vdots \\
 a_{m1} v_1 + \dots + a_{mn} v_n = -u_m
 \end{array} \right\} MV = -U$$

The systems are always dual in the sense of orthogonality, for

$$XU + YV = X(-MV) + (XM)V = 0 .$$

Definition 1: A column solution for the tableau is a real  $1 \times (m+n)$  matrix

$Z = (X, Y)$  such that  $XM = Y$ . A row solution is a real  $1 \times (m+n)$  matrix

$W = (U^T, V^T)$  such that  $MV = -U$ .

If variables and coefficients take values in some commutative field  $F$ , then assuming nondegeneracy, the column system generates an  $m$ -dimensional subspace and the row system an  $n$ -dimensional subspace of the  $(m+n)$ -space  $F^{m+n}$  determined by the vectors  $Z = (X, Y)$  and  $W = (U^T, V^T)$ . In an ordered field, the subspaces spanned by the row and column systems are not only orthogonal but also complementary in that the entire  $(m+n)$ -space is determined by their direct sum.

Since this paper is concerned with linear inequalities and notions of positivity, we necessarily restrict ourselves to ordered fields and therefore to complementary orthogonal systems. We specifically use the real numbers because applications such as matrix games require numerical solutions to practical problems. The results and methods are nevertheless easily generalized to arbitrary ordered fields.

Throughout this paper we use the symbols  $\oplus$  and  $\ominus$  respectively to denote nonnegative and nonpositive elements. Matrix inequalities hold componentwise. Thus  $M > 0$  means each component of  $M$  is positive,  $M \geq 0$  means each component of  $M$  is nonnegative, and  $M = 0$  means  $M$  is the



zero or trivial matrix. We adopt the special convention that  $M \geq 0$  means  $M \geq 0$  and  $M$  is nontrivial.

From the tableau it is easy to see that information about the sign patterns in the coefficient matrix  $M$  often provides a clue to the types of solutions we can expect to find. For example, if  $M$  has a nonnegative first row, then we can obtain a column solution  $Z \geq 0$  by setting  $x_1 > 0$  and  $x_i = 0$  for  $i > 1$ .

We temporarily restrict our investigations to linear systems with skew coefficient matrices over ordered fields, denoting such matrices by the letter  $A$ .

Definition 2: A matrix  $A$  is skew when  $A = -A^T$ . A skew matrix is necessarily square, and we say its order is  $n$  if it is  $n \times n$ . Over an ordered field the diagonal entries are always zero.

We present the tableau of a skew matrix in the following abbreviated form:

$$X \begin{array}{c} X^T \\ \boxed{A} \\ = Y \end{array} = -Y^T.$$

By negative transposition it is clear that any column solution  $(X, Y)$  is also a row solution when written  $(Y, X)$  and conversely. Hence for the skew case we are justified in dispensing with  $W = (U^T, V^T)$  as the row solution, and we say that the dual linear systems are isomorphic or that either system is self-dual.

Theorem 1: A matrix  $A$  of order  $n$  (over an ordered field) is skew if and only if  $XAX^T = 0$  for all  $1 \times n$  matrices  $X$ .

Proof: If  $A$  is skew, then for all  $1 \times n$  matrices  $X$  we have  $X(-A^T)X^T = XAX^T = (XAX^T)^T = XA^T X^T$ . Hence  $XAX^T = 0$ .

Conversely, suppose  $XAX^T = 0$  for all  $1 \times n$  matrices  $X$ . Given an integer  $i$ , choose  $X'$  such that  $x'_i = 1$  and  $x'_k = 0$  for  $k \neq i$ . Then for each  $i$ ,  $X'A(X')^T = a_{ii} = 0$ . Now given distinct integers  $i$  and  $j$ , choose  $X''$  such that  $x''_i = x''_j = 1$  and  $x''_k = 0$  for  $k \neq i, j$ . Then

$$\begin{aligned} 0 &= X''A(X'')^T = (a_{ii} + a_{jj}, \dots, a_{in} + a_{jn})(X'')^T \\ &= a_{ii} + a_{jj} + a_{ij} + a_{ji} \\ &= a_{ji} + a_{ij}. \end{aligned}$$

Hence  $a_{ji} = -a_{ij}$  for each  $i, j$  where  $1 \leq i \leq n$  and  $1 \leq j \leq n$ , which means  $A = -A^T$ . ■

We should like to transform a given tableau into another one with a different pattern of signs in the coefficient matrix while simultaneously maintaining the original solution sets, up to permutations of solution components. Exchanging some independent variable  $x_i$  for some dependent variable  $y_j$  will serve the purpose; for such an exchange corresponds precisely to solving a column equation for the chosen independent variable, substituting that expression into the remaining equations and simplifying the results. The analogous statement holds for the row system.

The quadratic form  $XAX^T$  associated with the skew matrix  $A$  is simply the inner product  $YX^T$ . Theorem 1 says  $YX^T = y_1x_1 + \dots + y_nx_n = 0$ .

Consequently we can preserve this inner product — and hence skewsymmetry in the matrix — while exchanging pairs of variables provided the exchange of  $x_i$  for  $y_j$  is concomitant with the exchange of  $x_j$  for  $y_i$ . Note that if  $i = j$  the variable  $x_i$  does not appear in the expression for  $y_i$  since its coefficient is  $a_{ii} = 0$ . The exchange operation can be defined only for pairs of variables where  $i \neq j$  and  $a_{ij} = -a_{ji} \neq 0$ .

Definition 3: A principal pair pivot or P-pivot in the tableau of a self-dual linear system is a simultaneous exchange of variables  $x_i$  with  $y_j$  and  $x_j$  with  $y_i$ , provided  $a_{ij} = -a_{ji} \neq 0$ , and a permutation that exchanges row  $i$  with row  $j$  and column  $i$  with column  $j$  while fixing the remaining rows and columns.

To see what happens in the skew tableau we consider an example of order  $n = 4$ .

Example 1: The tableau

	$x_1$	$x_2$	$x_3$	$x_4$	
$x_1$	0	$r$	$p^*$	$t$	$= -y_1$
$x_2$	$-r$	0	$q$	$s$	$= -y_2$
$x_3$	$-p^*$	$-q$	0	$u$	$= -y_3$
$x_4$	$-t$	$-s$	$-u$	0	$= -y_4$
	$= y_1$	$= y_2$	$= y_3$	$= y_4$	

represents a self-dual system of four linear equations:

$$(1) \quad -x_2 r - x_3 p - x_4 t = y_1$$

$$(2) \quad x_1 r - x_3 q - x_4 s = y_2$$

$$(3) \quad x_1 p + x_2 q - x_4 u = y_3$$

$$(4) \quad x_1 t + x_2 s + x_3 u = y_4$$

A P-pivot on the starred entries  $p$ ,  $-p \neq 0$  in the matrix corresponds to solving equations (3) and (1) for  $x_1$  and  $x_3$  respectively, substituting the resulting expressions into equations (2) and (4), simplifying the results and switching positions of equations (1) and (3) in both systems. Thus

$$x_3 = -y_1 p^{-1} - x_2 r p^{-1} - x_4 t p^{-1} \text{ from equation (1) ,}$$

and 
$$x_1 = y_3 p^{-1} - x_2 q p^{-1} - x_4 u p^{-1} \text{ from equation (3) .}$$

Substitution of these expressions into equations (2) and (4) and simplification yield:

$$\begin{aligned} y_2 &= (y_3 p^{-1} - x_2 q p^{-1} + x_4 u p^{-1})r - (-y_1 p^{-1} - x_2 r p^{-1} - x_4 t p^{-1})q - x_4 s \\ &= y_3 p^{-1} r + y_1 p^{-1} q - x_4 [s - p^{-1}(tq + ru)] \end{aligned}$$

and 
$$\begin{aligned} y_4 &= (y_3 p^{-1} - x_2 q p^{-1} + x_4 u p^{-1})t + x_2 s + (-y_1 p^{-1} - x_2 r p^{-1} - x_4 t p^{-1})u \\ &= y_3 p^{-1} t - y_1 p^{-1} u + x_2 [s - p^{-1}(tq + ru)] . \end{aligned}$$

The resulting new tableau is:

	$y_1$	$x_2$	$y_3$	$x_4$	
$y_1$	0	$p^{-1}q$	$-p^{-1}$	$-p^{-1}u$	$= -x_1$
$x_2$	$-qp^{-1}$	0	$-rp^{-1}$	$[s-p^{-1}(tq+ru)]$	$= -y_2$
$y_3$	$p^{-1}$	$p^{-1}r$	0	$p^{-1}t$	$= -x_3$
$x_4$	$up^{-1}$	$-[s-p^{-1}(tq+ru)]$	$-tp^{-1}$	0	$= -y_4$
	$= x_1$	$= y_2$	$= x_3$	$= y_4$	

Note that the marginal variables remain consecutively numbered. It is also evident that the tableau form with minus signs in the right-hand margin ensures that a single calculation suffices for both linear systems.

From the preceding example it is easy to generalize the notion of P-pivot to demonstrate its operation in arbitrarily large finite self-dual systems. Given the skew matrix  $A$ , a P-pivot on  $a_{ij}$ ,  $a_{ji} \neq 0$  in the tableau yields a skew matrix  $\bar{A}$  with the self-dual solution  $(\bar{X}, \bar{Y})$  such that  $\bar{x}_1 = y_1$ ,  $\bar{x}_j = y_j$ ,  $\bar{y}_1 = x_1$ ,  $\bar{y}_j = x_j$  and  $\bar{x}_k = x_k$ ,  $\bar{y}_k = y_k$  for all  $k \neq i, j$ . The new matrix  $\bar{A}$  is given by the following entries:

$$\bar{a}_{ij} = -a_{ij}^{-1} \text{ for } i \neq j$$

$$\bar{a}_{ih} = a_{ij}^{-1} a_{hj} \text{ for } h \neq i, j$$

$$\bar{a}_{kj} = -a_{ij}^{-1} a_{ik} \text{ for } k \neq i, j$$

$$\bar{a}_{kh} = a_{kh} - a_{ij}^{-1}(a_{ih} a_{kj} - a_{ki} a_{jh}) \text{ for } k \neq i, h \neq j$$

$$\bar{a}_{ii} = a_{ij}^{-1} a_{ii} = 0$$

To check that the described matrix transformation is in fact what occurs requires only elementary algebraic manipulations. The new matrix or tableau is called a P-pivot transform of the initial one. Note that the P-pivot transform has precisely the same solution set as the original tableau in the sense that values assigned to the variables in one tableau will hold for the same variables in the other even though positions of variables may have changed. Observe also that the P-pivot operation can be defined as above for the matrix alone, without reference to the marginal variables. For this reason we often say we P-pivot on  $a_{ij}$ ,  $a_{ji}$  in the matrix A.

We define one more operation on the tableau.

Definition 4: A principal permutation or P-permutation in a square tableau is a permutation of columns occurring simultaneously with the same permutation of the corresponding rows.

It is clear that P-permutations as well as P-pivots preserve both skew-symmetry and solution sets. We use the word "principal" in each case because the term is traditionally applied to a submatrix obtained by choosing certain rows and the correspondingly-labeled columns from a given matrix. We occasionally use the term P-steps to describe a sequence of P-permutations or P-pivots or both.

The following theorem is a consequence of the definitions.

Theorem 2: Up to P-permutations, the maximum number of (skew) matrices that can be obtained by P-pivots in a skew matrix of order  $n$  is  $2^{n-1}$ .

Proof: The net result of a finite sequence of P-pivots in the tableau corresponds precisely to choosing pairs of independent variables in the column system. Since we must pick an even number of variables, our choice of  $x_1$  depends on whether we have picked an odd number of  $x$ 's from the remaining  $n - 1$  possibilities. That is, the total set of choices for  $n$  variables is completely determined by the subsets of  $n - 1$  of the variables. Thus the number of choices is the number of subsets of an  $(n-1)$ -set, which is  $2^{n-1}$ .

It remains to show that a particular choice of independent variables corresponds within P-permutations to precisely one coefficient matrix. For each set  $X$  of independent variables and each integer  $i$ ,  $1 \leq i \leq n$ , we have a solution  $(X, Y)$  for which  $x_i = 1$ ,  $x_k = 0$  if  $k \neq i$ , and hence  $Y = (a_{i1}, \dots, a_{in})$ . Thus up to P-permutations any independent set of variables uniquely determines the rows of the matrix. ■

## §2. The Saddlepoint Theorem

Given a skew matrix  $A$  of order  $n$  over an ordered field, we are considering the finite class of skew matrices  $\bar{A}$  such that  $XA = Y$  and  $\bar{X}\bar{A} = \bar{Y}$  have the same solutions, where  $\bar{x}_i = y_i$  and  $\bar{y}_i = x_i$  for some indices  $i$  (perhaps none) and  $\bar{x}_i = x_i$ ,  $\bar{y}_i = y_i$  for the remaining indices  $i$ ,  $1 \leq i \leq n$ .

Definition 5: [7] A nontrivial solution  $(X, Y)$  of  $XA = Y$  is elementary if the only other solutions having the same zero components as  $(X, Y)$  are of the form  $(kX, kY)$  for scalars  $k \neq 0$ .

Lemma 1: Let  $A$  be a skew matrix of order  $n$  and let  $h$  be an index,  $1 \leq h \leq n$ . There exists a nonnegative elementary solution of  $XA = Y$  such that  $x_h + y_h > 0$  if and only if there exists a skew matrix  $\bar{A}$  having the same solutions as  $A$  and such that  $\bar{a}_{hj} \geq 0$  for all  $j$ ,  $1 \leq j \leq n$ .

Proof: Given  $h$ , suppose there exists a nonnegative elementary solution of  $XA = Y$  such that  $x_h + y_h > 0$ . Then either  $x_h = 0$  and  $y_h > 0$  or  $x_h > 0$  and  $y_h = 0$ . If  $a_{hj} \geq 0$  for all  $j$ , there is nothing more to do. We therefore assume there is at least one index  $j \neq h$  such that  $a_{hj} < 0$ , giving the schema:

$x_h$	0	-
	+	
	$=y_h$	

Suppose  $x_h = 0$  and  $y_h > 0$ . Then there exists an index  $t \neq h$  such that  $x_t > 0$  and  $a_{th} > 0$ . If  $x_t$  can become arbitrarily large without decreasing any  $y_j$ , then we must have  $a_{tj} \geq 0$  for all  $j$ ,  $1 \leq j \leq n$ . In that case a P-pivot on  $a_{th}$ ,  $a_{ht}$  yields an  $\bar{A}$  with  $\bar{a}_{hj} \geq 0$  for all  $j$ . Otherwise we can increase  $x_t$  until we obtain  $y_k = 0$  for some  $a_{tk} < 0$  and  $k \neq h$ ,



with  $y_j \geq 0$  for  $j \neq k$ . A P-pivot on  $a_{tk}$ ,  $a_{kt}$  will exchange  $x_t$  with  $y_k$  and  $x_k$  with  $y_t$ . Even if  $x_k = 0$  the P-pivot will increase the number of zeros in the left margin by at least one because  $x_t > 0$  implies  $y_t = 0$ , and already  $y_k = 0$ . In fewer than  $n$  steps we reach the form

0	$\begin{matrix} \cdot & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{matrix}$		$\ominus$	
$\cdot$			$\vdots$	
$\cdot$				
$\cdot$				
$\cdot$				
0				
+	$\oplus \dots$	+	$\dots$	$\oplus$
0				
$\vdots$				
0				
	$=\oplus \dots$	$=+$	$\dots$	$=\oplus$

A P-pivot on the starred entries yields an  $\bar{A}$  with  $\bar{a}_{hj} \geq 0$  for all  $j$ .

Suppose alternatively that  $x_h > 0$  and  $y_h = 0$ . A P-pivot on some  $a_{hl} < 0$  and  $a_{lh} > 0$  yields an  $\bar{A}$  with  $\bar{x}_h = 0$  and  $\bar{y}_h > 0$ , which is the previous case.

To prove the converse, suppose there exists a skew matrix  $\bar{A}$  having the same solutions as  $A$  and such that  $\bar{a}_{hj} \geq 0$  for all  $j$ ,  $1 \leq j \leq n$ . Then we have the following tableau:

	$\ominus$	
	$\vdots$	
$\bar{x}_h$	$\oplus \dots 0 \dots \oplus$	
	$\vdots$	
	$\ominus$	
	$=\bar{y}_h$	

Setting  $\bar{x}_i = 0$  for  $i \neq h$  and  $\bar{x}_h > 0$  gives a solution  $(\bar{X}, \bar{Y}) \geq 0$  such that  $\bar{x}_h > 0$  and  $\bar{y}_h = 0$ . Either  $\bar{x}_h = x_h > 0$  and  $\bar{y}_h = y_h = 0$  or  $\bar{x}_h = y_h > 0$  and  $\bar{y}_h = x_h = 0$ . In either case  $x_h + y_h > 0$  and  $x_i + y_i \geq 0$  for all  $i \neq h$ , where  $XA = Y$ . The nonnegative solution  $(X, Y)$  is elementary because it is determined by the entries  $\bar{a}_{hj} \geq 0$  up to scalar multiples dependent on the value of  $\bar{x}_h$ . ■

We can now prove the main existence theorem.

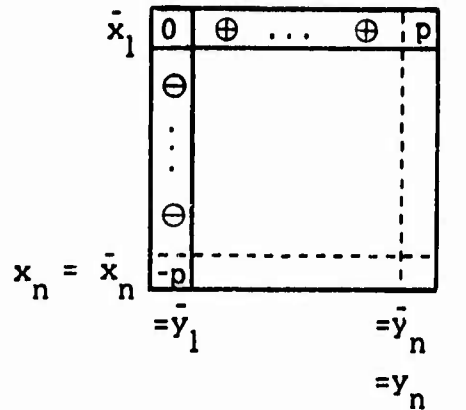
**Theorem 3:** Given a skew matrix  $A$  of order  $n$  and an index  $h$ ,  $1 \leq h \leq n$ , there exists a nonnegative elementary solution of  $XA = Y$  such that  $x_h + y_h > 0$ , and there exists a skew matrix  $\bar{A}$  having the same solutions as  $A$  and such that  $\bar{a}_{hj} \geq 0$  for all  $j$ ,  $1 \leq j \leq n$ .

**Proof:** We use induction on the order  $n$  of  $A$ . The theorem is clearly true for any trivial  $A$  and for  $n = 1, 2$ .

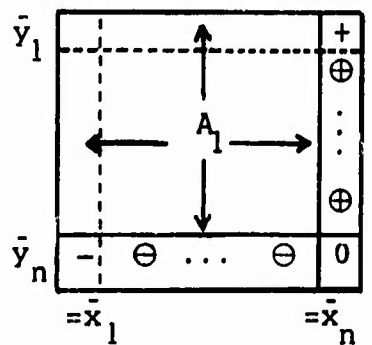
$$n = 1: \quad + \begin{array}{|c|} \hline 0 \\ \hline \end{array} \\ \quad \quad \quad = 0$$

$$n = 2: \quad + \begin{array}{|cc|} \hline 0 & + \\ \hline 0 & - \\ \hline \end{array} \begin{array}{c} \text{P-pivot} \\ \longleftrightarrow \end{array} \begin{array}{c} 0 \\ + \end{array} \begin{array}{|cc|} \hline 0 & - \\ \hline + & 0 \\ \hline \end{array} \\ \quad \quad \quad = 0 \quad = + \quad \quad \quad = + \quad = 0$$

As induction hypothesis we assume the theorem is true for skew matrices of order  $k < n$ . Without loss of generality we choose  $h = 1$ . That means for a skew matrix  $A$  of order  $n$  we can reach the following form:



Either  $p \geq 0$  and the theorem is satisfied, or  $p < 0$ . For  $p < 0$  a P-pivot on  $p, -p$  yields:



By the induction hypothesis there is a nonnegative elementary solution for the skew submatrix  $A_1$  such that  $\bar{x}_1 + \bar{y}_1 > 0$ . Setting  $\bar{y}_n = 0$  extends the solution to one for  $A$  of the desired form. Applying Lemma 1 completes the proof. ■

**Definition 6:** A zero saddlepoint in a matrix is a sign configuration consisting of a nonnegative row and a nonpositive column. The entry forming their intersection is necessarily zero.

In skew matrices any nonnegative row or nonpositive column determines a zero saddlepoint with the correspondingly-labeled column or row. In view

of Definition 6, we call Theorem 3 the Saddlepoint Theorem. It can also be derived as a special case of a theorem on positive semidefinite matrices in T. D. Parsons' thesis [6, pp. 84-85]; but our proof is both elementary and entirely self-contained.

Corollary 1: [Tucker Skew-Symmetric Matrix Theorem]

If  $A$  is a skew matrix, then there exists a column solution  $Z = (X, Y) \geq 0$  to its tableau such that  $X + Y > 0$  and  $YX^T = 0$ .

Proof: Given a skew matrix  $A$  of order  $n$ , Theorem 3 says that for each index  $i$ ,  $1 \leq i \leq n$ , there exists a solution  $Z_i = (X, Y) \geq 0$  to the matrix tableau such that either  $x_i > 0$  or  $y_i > 0$ . Because the tableau represents linear systems, the sum of solutions of the type  $Z_i$  for a given tableau yields another solution

$$\bar{Z} = (\bar{X}, \bar{Y}) = \sum_{i=1}^n Z_i \geq 0$$

such that either  $\bar{x}_i > 0$  or  $\bar{y}_i > 0$  for each  $i$ ,  $1 \leq i \leq n$ . By Theorem 1,  $YX^T = 0$  for any skew tableau, and the corollary is proved. ■

The Skew-Symmetric Matrix Theorem appeared in 1956 [10]. Goldman and Tucker [3] first indicated its importance in a paper describing some of the theory of linear programming for which it is the cornerstone result.

Corollary 2: [Tucker Complementary Slackness Theorem]

Let  $M$  be an  $m \times n$  matrix over an ordered field. There exist solutions to the tableau

$$X \begin{array}{c} V \\ \boxed{M} \\ =Y \end{array} = -U$$

such that  $(X, Y) \geq 0$ ,  $(U^T, V^T) \geq 0$ ,  $X + U^T > 0$ ,  $V^T + Y > 0$  and  $XU + YV = 0$ .

Proof: Given  $M$ , we form its  $n \times m$  negative transpose and two trivial square matrices of orders  $m$  and  $n$  to construct the expanded tableau  $A$ :

$$\begin{array}{c} x_1 \dots x_m \quad v_1 \dots v_n \\ \begin{array}{cc|c} x_1 & & = -u_1 \\ \vdots & & \vdots \\ x_m & 0 & M & = -u_m \\ \hline v_1 & -M^T & 0 & = -y_1 \\ \vdots & & \vdots & \vdots \\ v_n & & & = -y_n \end{array} \\ =u_1 \dots =u_m =y_1 \dots =y_n \end{array} \quad \text{or} \quad \begin{array}{cc|c} X^T & V & \\ \hline X & 0 & M & = -U \\ \hline V^T & -M^T & 0 & = -Y \\ \hline =U^T & =Y & \end{array}$$

where  $X$  is  $1 \times m$ ,  $Y$  is  $1 \times n$ ,  $V$  is  $n \times 1$  and  $U$  is  $m \times 1$ . The labeling of variables in the tableau is justified by the observation that in general any column (row) solution to the tableau of  $M$  is a row (column) solution to the tableau of  $-M^T$  and vice versa.

Clearly  $A = -A^T$ , so the expanded matrix is skew. By Corollary 1 there exists a column solution  $(X, V^T, U^T, Y) \geq 0$  to  $A$  such that  $(X, V^T) + (U^T, Y) > 0$  and  $(U^T, Y)(X, V^T)^T = 0$ . In particular,  $\lambda + U^T > 0$  and  $V^T + Y > 0$ ; also  $U^T X^T = 0$  and  $YV = 0$ , and the proof is complete. ■

Corollary 1 is clearly a self-dual form of Corollary 2. In fact the results are equivalent because we can derive Corollary 1 from Corollary 2 simply by observing that in the self-dual tableau,  $X = V^T$  and  $Y = U^T$ .

The notion of complementary slackness in dual linear systems was introduced with the Complementary Slackness Theorem by Tucker in 1956 [10] and developed further by him in 1968 [14]. The idea is that in each complementary pair of variables  $x_i$  and  $u_i$  or  $v_j$  and  $y_j$ , precisely one of them is "slack" — that is, strictly positive.

### §3. Application to Matrix Games

Using the results of the previous section, we give a direct elementary proof of John von Neumann's Minimax Theorem [15, p. 153] for the special case of symmetric games.

Any real matrix  $G = (g_{ij})$  determines a two-person game in which one player chooses a row  $i$  while the other chooses a column  $j$ , and the column player pays the row player the amount  $g_{ij}$ . If  $g_{ij} < 0$  then the row player gives the column player the payoff  $|g_{ij}|$ . Such a game is called zero-sum because the relation

$$\text{row player's winnings} + \text{column player's winnings} = 0$$

holds for each play or pair of choices consisting of a row and a column. It is clear that in any matrix game the row player is trying to maximize his gains at the same time that the column player attempts to minimize his own losses.

**Definition 7:** A solution to an  $m \times n$  matrix game  $G$  is an  $(m+n+1)$ -tuple  $[p_1, \dots, p_m; q_1, \dots, q_n; v] = [P; Q^T; v]$  such that

$$(1) \quad \sum_{i=1}^m p_i = \sum_{j=1}^n q_j = 1, \quad p_i \geq 0 \text{ and } q_j \geq 0.$$

$$(2) \quad \sum_{i=1}^m p_i g_{ij} \geq v, \quad 1 \leq j \leq n.$$

$$(3) \quad \sum_{j=1}^n g_{ij} q_j \leq v, \quad 1 \leq i \leq m.$$

Condition (1) says simply that  $P = (p_1, \dots, p_m)$  and  $Q^T = (q_1, \dots, q_n)$  are probability vectors. Conditions (2) and (3) correspond to the following tableau:

$$\begin{array}{c} \begin{array}{c} q_1 \quad \dots \quad q_n \\ \hline \begin{array}{c} p_1 \\ \vdots \\ p_m \end{array} \begin{array}{c} \boxed{G} \\ \hline \end{array} \end{array} \begin{array}{c} \leq v \\ \vdots \\ \leq v \\ \hline \geq v \quad \dots \quad \geq v \end{array} \end{array}$$

The number  $v$ , which von Neumann showed exists uniquely for any game, is called the value of the game. If  $v = 0$  the game is called fair.

The vectors  $P$  and  $Q$  are the respective mixed strategies for the row and column players. If the row player chooses row  $i$  with probability  $p_i$ , he can expect to gain at least  $v$  regardless of what the column player does. Similarly the column player can expect to lose at most  $v$  if he chooses column  $j$  with probability  $q_j$ .

Definition 8: A matrix game is symmetric if its payoff matrix is skew.

From the definition it is clear that the roles of row and column players are interchangeable in symmetric games. For a broader definition that includes other types of matrices see Shapley [8].

Theorem 4: [von Neumann Minimax Theorem for Symmetric Games]

Every two-person zero-sum symmetric game has a solution  $[P; Q^T; v]$  for which  $v = 0$  and  $P = Q^T$ .

Proof: We form the tableau for skew  $G$  :

$$\begin{array}{c}
 \begin{array}{cc}
 & \begin{array}{ccc} q_1 & \dots & q_n \end{array} \\
 \begin{array}{c} p_1 \\ \vdots \\ p_n \end{array} & \boxed{\begin{array}{c} G \end{array}} \\
 & \begin{array}{ccc} = y_1 & \dots & = y_n \end{array}
 \end{array}
 \begin{array}{c}
 = -u_1 \\
 \vdots \\
 = -u_n
 \end{array}
 \quad \text{or} \quad
 \begin{array}{c}
 \begin{array}{cc}
 & Q \\
 P & \boxed{\begin{array}{c} G \end{array}} \\
 & = Y
 \end{array}
 = -U
 \end{array}
 \end{array}$$





$$c \leq \min_j y_j = (\min_j y_j) \sum_{j=1}^n v_j \leq YV = -XU \leq -\sum_{i=1}^m x_i (\min_i u_i) = -(\min_i u_i) \leq d \quad \blacksquare$$

**Theorem 5:** [The von Neumann Minimax Theorem]

Every two-person zero-sum matrix game has a solution and a unique value.

Proof: [13] If  $G$  is an  $m \times n$  matrix game we form an  $(m+n+2) \times (m+n+2)$  skew tableau  $A$  as follows:

	$p_1 \dots p_m$	$q_1 \dots q_n$	$r$	$t$		
$p_1$	0	G	-1	1	$= -y_1$	
$\vdots$			$\vdots$	$\vdots$		$\vdots$
$p_m$			-1	1		
$q_1$	$-G^T$	0	1	-1		
$\vdots$			$\vdots$	$\vdots$		$\vdots$
$q_n$						1
$r$	1 ... 1	-1 ... -1	0	0		
$t$	-1 ... -1	1 ... 1	0	0	$= -y_{m+n+2}$	

$= y_1 \dots \dots \dots = y_{m+n+2}$

where  $P = (p_1, \dots, p_m)$

$Q^T = (q_1, \dots, q_n)$

$Y = (y_1, \dots, y_{m+n+2})$

By Corollary 1 there exists a solution  $(P, Q^T, r, t, Y) \geq 0$  to the self-dual system of  $A$  such that  $(P, Q^T, r, t) + Y > 0$  and  $Y(P, Q^T, r, t)^T = 0$ . The last two column equations of the tableau yield:

$$0 \leq y_{m+n+2} = \sum_{i=1}^m p_i - \sum_{j=1}^n q_j = -y_{m+n+1} \leq 0 .$$

Hence  $y_{m+n+1} = y_{m+n+2} = 0$ ,  $r > 0$ ,  $t > 0$  and  $\sum_{i=1}^m p_i = \sum_{j=1}^n q_j$ . Then either  $P = 0$  and  $Q^T = 0$  or  $P \geq 0$  and  $Q^T \geq 0$ .

If  $P = Q^T = 0$ , then  $y_j > 0$  for  $1 \leq j \leq m+n$ . But  $y_1 = r - t = -y_{m+n+1}$ , which would be impossible. Hence  $P \geq 0$  and  $Q^T \geq 0$ . Let  $s = \sum_{i=1}^m p_i = \sum_{j=1}^n q_j > 0$

and  $v = s^{-1}(r-t)$ . Then  $[s^{-1}P; s^{-1}Q^T; v]$  is a solution to  $G$ ; for

$$\sum_{i=1}^m s^{-1}p_i = \sum_{j=1}^n s^{-1}q_j = 1 \text{ and } s^{-1}PG \geq s^{-1}(r-t) \geq s^{-1}Q^TG^T = Gs^{-1}Q .$$

To show uniqueness of  $v$ , assume  $[P_1; Q_1^T; v_1]$  and  $[P_2; Q_2^T; v_2]$  are distinct solutions of the game. Then  $v_1 \leq P_1G$  for one column solution of the tableau and  $GQ_2 \leq v_2$  for one row solution, thus satisfying Lemma 2. Hence  $v_1 \leq v_2$ . Similarly  $v_2 \leq v_1$ , and therefore  $v_1 = v_2$ . ■

A survey of alternative methods of proving the Minimax Theorem appears in the article by Kuhn and Tucker on von Neumann's work in game theory [5]. A paper by Gale, Kuhn and Tucker [2] contains further details on other methods of skewsymmetrization that lead to algebraic proofs. For an efficient algorithm that uses linear programming to solve matrix games, see Tucker [11].

Adopting the terminology of Gale [1] we can make fuller use of the complementary slackness principle in solutions of arbitrary  $m \times n$  matrix games. Each choice of a single column or row constitutes a pure strategy for the

appropriate player. The probability vectors in a solution of the game constitute optimal mixed strategies in that they specify the frequency with which a particular pure strategy should be chosen in order for the player to minimize his expected losses or maximize his expected gains. Gale's idea is to partition the probability vectors for a given solution of the game into their positive and zero components and to call the corresponding pure strategies essential and superfluous.

Thus if we have a solution  $[P; Q^T; v]$  to the  $m \times n$  game  $G$  we can obtain the following tableau by permuting rows and columns:

	$q_1$	$\dots$	$q_h$	$q_{h+1}$	$\dots$	$q_n$	
	+	$\dots$	+	0	$\dots$	0	
$p_1 = +$	$G_1$		$(k \times h)$				$= -v$
$\vdots$							$\vdots$
$p_k = +$							$= -v$
$p_{k+1} = 0$							$< -v$
$\vdots$							$\vdots$
$p_m = 0$							$< -v$
	$= v$	$\dots$	$= v$	$> v$	$\dots$	$> v$	

The rows  $k + 1, \dots, m$  are "superfluous" strategies in the sense that if the row player chooses one of them while the column player sticks to his "essential" choices  $1, \dots, h$  then in time the payoff to the row player will be less than

the value  $v$  which he could have obtained had he stuck to his own "essential" choices. An analogous statement holds for the "superfluous" strategies  $h + 1, \dots, n$  of the column player. Of course, the set of superfluous strategies for either player may be empty.

In a symmetric game we can always find a symmetric solution — that is, one where  $p_i = q_i$  for  $1 \leq i \leq n$ . For such solutions the permutation to display the partition of mixed strategies will be a P-permutation and will yield a principal square submatrix  $G_1$  of  $G$  corresponding to the essential strategies.

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